Factorial Notation

A product of the form $x(x-1)(x-2)\ldots(x-n+1)$ is denoted by $x^{(n)}$ and is called a factorial or a factorial polynomial of degree 'n'.

i.e., $x^{(1)} = x$

$x^{(2)} = x(x-1)$

$x^{(3)} = x(x-1)(x-2)$

... ... and so on.

If the interval difference is 'h' then $x^{(n)} = x(x-h)(x-2h)\ldots(x-(n-1)h)$

Difference of a Factorial Polynomial

Statement: If the interval of differencing is 'h', then $\Delta^n x^{(n)} = n!\,h^n$ and $\Delta^{n+1} x^{(n)} = 0$

Proof: By definition of factorial polynomial, we have,

$$x^{(n)} = x(x-h)(x-2h)\ldots(x-(n-1)h) \quad \text{--- (1)}$$

so that $(x+h)^{(n)} = (x+h)x(x-h)(x-2h)\ldots(x-(n-2)h)$ --- (2)

Now, $\Delta x^{(n)} = (x+h)^{(n)} - x^{(n)}$

$$= (x+h)x(x-h)(x-2h)\ldots(x-(n-2)h) - x(x-h)(x-2h)\ldots(x-(n-1)h)$$

$$= x(x-h)(x-2h)\ldots(x-(x-2)h)[x+h-(x-(n-1)h)]$$

$$= x(x-h)(x-2h)\ldots(x-(n-1)h)nh$$

$$\Rightarrow \Delta x^{(n)} = nhx^{(n-1)} \quad \text{--- (3)}$$

RHS of ③ is a factorial polynomial degree (n-1)

Again $\Delta^2 x^{(n)} = [\Delta x^{(n)}] = \Delta[nh\, x^{(n-1)}]$ — from ③

$= nh\,\Delta x^{(n-1)} = nh[(x+h)^{(n-1)} - x^{(n-1)}]$

$= nh[\{(x+h)x(x-h)\dots(x-(n-3)h\}$

$-\{x(x-h)(x-2h)\dots(x-(n-2)h)\}]$

$= nh[x(x-h)\dots(x-(x-3)h)\{x+h-(n-2)h\}]$

$= nh[x(x-h)\dots(x-(x-3)h)\{(n-1)h\}]$

$= n(n-1)h^2 x(x-h)\dots(x-(n-3)h)$

$\Delta^2 x^{(n)} = n(n-1)h^2 x^{(n-2)}$ ——— ④

RHS of ④ is a factorial polynomial of degree (n-2)

Continuing in this way upto 'n' times, we get

$\Delta^n x^{(n)} = n(n-1)(n-2)\dots 2.1\, h^n x^{(n-n)}$

$= n!\, h x^{(0)} = n!\, h^n$

$\Rightarrow \Delta^n x^{(n)} = n!\, h^n$ ——— ⑤

Proved

RHS of ⑤ is a constant, hence

$\Delta^{n+1} x^{(n)} = \Delta(\Delta^n x^{(n)}) = \Delta(n!\, h^n) = 0$

$\Rightarrow \Delta^{n+1} x^{(n)} = 0$

Proved

★ Note : 1) If $h=1$, $\Delta^n x^{(n)} = n!$

2) From eqn ③ we observe $\Delta x^{(n)} = n x^{(n-1)}$ for $h=1$, and this resembles with $\Delta x^n = n x^{n-1}$. Hence, the operator Δ behaves in a same way on factorial polynomial as operator Δ behave on ordinary polynomial.

Prove that $x^{(-n)} = \frac{1}{(x+n)^{(n)}}$, if $h=1$

Proof: By the definition of $x^{(n)}$ for $h=1$ we have $x^{(n)} = x(x-1)(x-2)\dots[x-(n-1)]$

$$\Rightarrow x^{(n)} = x^{(n-1)}(x-n+1) \quad \text{—— (1)}$$

Put $n=0$ we get

$$x^{0} = x^{(-1)}(x+1)$$

$$\Rightarrow x^{-1} = \frac{x^{(0)}}{x+1} = \frac{1}{(x+1)} = \frac{1}{(x+1)^{(1)}} \quad \text{—— (2)}$$

Put $n=-1$ in (1), we get

$$x^{-1} = x^{(-2)}(x+2)$$

$$\Rightarrow x^{(-2)} = \frac{x^{(-1)}}{(x+2)} = \frac{1}{(x+1)(x+2)}$$

$$\Rightarrow x^{(-2)} = \frac{1}{(x+2)^{(2)}} \quad \text{—— (3)}$$

Similarly, for $n=-2$

$$x^{(-3)} = \frac{1}{(x+3)(x+2)(x+1)} = \frac{1}{(x+3)^{(3)}}$$

$$\dots\dots\dots\dots\dots$$

$$x^{(-n)} = \frac{1}{(x+n)(x+n-1)\dots(x+1)}$$

$$\Rightarrow x^{(-n)} = \frac{1}{(x+n)^{(n)}}$$

Proved

In general if the interval of differencing is 'h', then

$$x^{(-n)} = \frac{1}{(x+h)(x+2h)\dots(x+nh)} = \frac{1}{(x+nh)^{(n)}} \quad \text{—— (4)}$$

Now, $\Delta x^{(-n)} = \Delta\left[\frac{1}{(x+nh)^{(n)}}\right]$

$$= \frac{1}{(x+(n+1)h)^{(n)}} - \frac{1}{(x+nh)^{(n)}}$$

$$= \frac{1}{(x+(n+1)h)(x+nh)\ldots(x+2h)}$$

$$- \frac{1}{(x+nh)\ldots(x+2h)(x+h)}$$

$$= \frac{1}{(x+nh)\ldots(x+2h)}\left[\frac{1}{x+(n+1)h} - \frac{1}{x+h}\right]$$

$$= \frac{(x+h)-(x+(n-1)h)}{(x+h)(x+2h)\ldots(x+(n+1)h)}$$

$$\Rightarrow \Delta x^{(-n)} = \frac{-nh}{[x+(n+1)h]^{(n+1)}} = -nh\,x^{(-(n+1))} \text{ --- } (5)$$

(from (4))

$$\Rightarrow \frac{\Delta}{\Delta x} x^{(-n)} = (-n)\,x^{(-(n+1))} \quad \text{---} (\because h = \Delta x)$$

Similarly, $\Delta^2 x^{(-n)} = \Delta[\Delta x^{(-n)}] = \Delta[-nh\,x^{(-(n+1))}]$

$$= -nh\,\Delta x^{(-(n+1))}$$

$$= (-nh)[-(n+1)h]\,x^{(-(n+2))} \text{ --- using (5)}$$

$$\Rightarrow \Delta^2 x^{(-n)} = (-1)^2 n(n+1)h^2 x^{(-n(n+2))}$$

$$\Rightarrow \frac{\Delta^2}{\Delta x^2} = (-1)^2 n(n+1)\,x^{(-(n+1))}$$

Proved